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On the dimensions of bivariate spline spaces and the stability of the dimensions ☆

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ABSTRACT

In this paper, the dimensions of bivariate spline spaces are studied using the Smoothing Cofactor-Conformality method. Based on the analysis on the conformality condition at one interior vertex, the stability (or singularity to the contrary) of the dimensions of general spline spaces is discussed in detail. By the aid of directed partition some new results on dimensions are obtained with the corresponding constraints depending on the degree, the smoothness order of the spline spaces and the structure of the partition as well.
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1. Introduction

Let D be a domain in $\mathbb{R}^2$ and Δ be a partition of D consisting of finite irreducible algebraic curves. Denote by $D_i, i = 1, 2, \ldots, N$, all the cells of Δ. For integers $k > \mu \geq 0$, we define

$$S_k^\mu(\Delta) := \{s \in C^\mu(D) : s|_{D_i} \in \mathbb{P}_k, i = 1, 2, \ldots, N\}$$

the bivariate spline space with degree k and smoothness order μ, where $\mathbb{P}_k$ denotes the space of all bivariate polynomials of total degree at most k.

The dimension of a spline space is important for the theories and applications on multivariate splines. However, the problem of determining the dimension of $S_k^\mu(\Delta)$ is difficult since it heavily depends on the geometric properties of the partition. For a regular triangulation Δ of D, the lower and upper bounds on the dimension of spline spaces given in [1,2] were far apart for large μ. When $k \geq 4\mu + 1$, the dimension of the spline space was actually given via the lower bound in [3] and it was given in [4] when $k \geq 3\mu + 2$. In general, the dimension of a spline space $S_k^\mu(\Delta)$ is singular in many cases. For example, let Δ_{ms} denote a Morgan–Scott partition [5] as shown in Fig. 1. Shi [6] proved that

$$\dim S_2^1(\Delta_{ms}) = \begin{cases} 7, & \text{when } A_1A_1', A_2A_2', A_3A_3' \text{ intersect at one point,} \\ 6, & \text{otherwise.} \end{cases}$$

The instability of the dimension of the spline space on the Morgan–Scott partition was studied further in [7–9].

Those results on the dimensions were obtained with the conditions depending only on k and μ, i.e., the degree and the smoothness order of the spline spaces. In fact, the structure of the partition is also a key involved in the dimension of the spline space. In [10,11], we studied the spline spaces on T-mesh and quasi-rectangular partition. The dimension formulas were derived with certain constraints depending on the structure of the partition.

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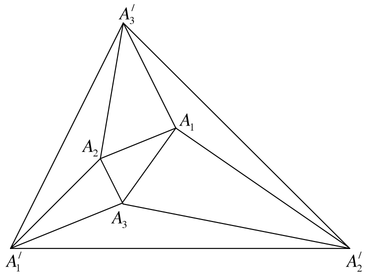


Fig. 1. A Morgan–Scott partition.

In this paper, the dimensions of the spline spaces on arbitrary partitions and the singularity of the dimensions are further discussed using the Smoothing Cofactor-Conformality method. Some new results are obtained with the corresponding constraints depending on the degree, the smoothness order and the structure of the partition as well. The rest of the paper is organized as follows. Section 2 discusses the stability and weak stability of the dimension of the spline space based on the analysis on the conformality condition at one interior vertex. The main results of the dimensions are presented in Section 3 by the aid of directed partition. The applications to the bivariate spline space on the Morgan–Scott partition are demonstrated in Section 4.

2. Stability and weak stability of the dimension of the spline space

At first, we review some basic results on multivariate spline by the Smoothing Cofactor-Conformality method. More details can be found in [12,13].

We only consider that the partition Δ consists of finite straight line segments throughout the present paper, such as (quasi-) crosscut partition, triangulation, quadrangulation. Except the boundary of the domain (denoted by ∂D), each straight line segment in Δ is called an interior grid line. Hence, there are three kinds of interior grid lines in Δ.

(a) *crosscut*: both of its endpoints lie on ∂D.
(b) *ray*: only one of its endpoints lies on ∂D.
(c) *Truncated-line (or T-line)*: neither of its endpoints lie on ∂D.

We call an intersection point of the interior grid lines as an interior vertex, and the intersection point of several crosscuts as a crosscutting vertex, especially. Besides, we call a partition as crosscut partition (denoted by Δ_c), if all the grid lines are crosscuts, and a partition as quasi-crosscut partition (denoted by Δ_{qc}), if each grid line is either a crosscut or a ray.

Let $(\alpha_1, \beta_1), \ldots, (\alpha_n, \beta_n)$ be pairwise linearly independent ordered pairs, that is, $\alpha_i\beta_j \neq \alpha_j\beta_i$ for $i \neq j,\ i, j = 1, \ldots, n$, and V_n be the vector space corresponding to the conformality condition at the origin O:

$$V_n = \left\{ (q_1, \ldots, q_n) : \sum_{i=1}^{n} q_i(x, y)(\alpha_i x + \beta_i y)^{\mu+1} \equiv 0,\ q_1, \ldots, q_n \in \mathbb{P}_{k-\mu-1} \right\}. \tag{1}$$

Denote by $d_k^\mu(n)$ the dimension of V_n. Two equivalent formula of $d_k^\mu(n)$ were presented in [14,1] as follows

$$\begin{aligned} d_k^\mu(n) &= \frac{1}{2}\left(k - \mu - \left[\frac{\mu+1}{n-1}\right]\right)_+ \times \left((n-1)k - (n+1)\mu + (n-3) + (n-1)\left[\frac{\mu+1}{n-1}\right]\right) \\ &= \sum_{j=0}^{k-\mu-1} (n(k-\mu-j) - (k+1-j))_+, \end{aligned} \tag{2}$$

where $[x]$ denotes the largest integer $\leq x$, $u_+ = \max\{0, u\}$.

By the Smoothing Cofactor-Conformality method, the dimension of a spline space $S_k^\mu(\Delta)$ is determined by the system of the global conformality conditions at all interior vertices of Δ. Since the conformality condition at each interior vertex of crosscut partition Δ_c or quasi-crosscut partition Δ_{qc} can be solved independently, the dimensions of spline spaces on Δ_c or Δ_{qc} were obtained as follows.

Theorem 1 ([14]). *Let D be a simply connected domain in $\mathbb{R}^2$ and Δ_c be a crosscut partition of D with L crosscuts and V interior vertices $A_1, \ldots, A_V$ in D such that e_i crosscuts intersect at A_i, $i = 1, \ldots, V$. Then the dimension of the bivariate spline space $S_k^\mu(\Delta_c)$, $0 \leq \mu \leq k-1$, is*

$$\dim S_k^\mu(\Delta_c) = \binom{k+2}{2} + L\binom{k-\mu+1}{2} + \sum_{i=1}^{V} d_k^\mu(e_i). \tag{3}$$

Theorem 2 ([14]). *Let D be a simply connected domain in $\mathbb{R}^2$ and Δ_{qc} be a quasi-crosscut partition of D with L_1 crosscuts, L_2 rays and V interior vertices $A_1, \ldots, A_V$ in D such that e_i crosscuts and rays intersect at A_i, $i = 1, \ldots, V$. Then the dimension of the bivariate spline space $S_k^\mu(\Delta_{qc})$, $0 \le \mu \le k-1$, is*

$$\dim S_k^\mu(\Delta_{qc}) = \binom{k+2}{2} + L_1 \binom{k-\mu+1}{2} + \sum_{i=1}^{V} d_k^\mu(e_i). \tag{4}$$

However, if there are T-lines in the partition, then the conformality conditions at different interior vertices are dependent on each other. That is a key and one of the difficulties dealing with the problems in multivariate splines.

As mentioned above, the dimension of $S_k^\mu(\Delta)$ heavily depends on the geometric properties of the partition. In fact, there are some differences in the singularity of the dimension according to different geometric quantities of the partition. In this section, the differences will be discussed based on the dimension formula $d_k^\mu(n)$ of the conformality condition at one interior vertex.

According to the graph theory, an arbitrary partition Δ can be treated as a graph. Then each line segment connecting two adjacent vertices on a grid line is called an edge, and the number of edges intersecting at an interior vertex A_i is called the degree of the vertex, denoted by $\deg A_i$. Let e_i denote the number of edges with different slopes intersecting at A_i, and c_i denote the number of grid lines crosscutting at A_i. Then $\deg A_i = e_i + c_i$.

Definition 1. Given a spline space $S_k^\mu(\Delta)$ defined on a partition Δ including V_I interior vertices, for each interior vertex A_i, $N_i = \deg A_i$, and e_i denotes the number of edges with different slopes intersecting at A_i, $i = 1, 2, \ldots, V_I$. We define

(1) the dimension of $S_k^\mu(\Delta)$ is (strongly) stable if it depends only on k, μ and N_i ($i = 1, 2, \ldots, V_I$);
(2) the dimension of $S_k^\mu(\Delta)$ is weakly stable if it depends only on k, μ and e_i ($i = 1, 2, \ldots, V_I$).

It is clear that the dimension of the spline space defined on a partition without interior vertex is stable. By Definition 1, the dimension of the spline spaces on crosscut partitions or quasi-crosscut partitions are weakly stable, and the dimension of the spline space $S_2^1(\Delta_{ms})$ on the Morgan–Scott partition is not weakly stable at all.

Note that the construction of the conformality condition V_n at one interior vertex is fundamental to the dimensions of the spline spaces. For example, $d_3^1(n)$ is the dimension of the solution space of the linear system: $\sum_{i=1}^n (a_i x + b_i y + c_i)(\alpha_i x + \beta_i y)^2 = A \cdot c = 0$, where $c = (a_1, b_1, \ldots, a_n, b_n, c_1, \ldots, c_n)^T$, and

$$A = \begin{pmatrix} \alpha_1^2 & 0 & \cdots & \alpha_n^2 & 0 & 0 & \cdots & 0 \\ 2\alpha_1\beta_1 & \alpha_1^2 & \cdots & 2\alpha_n\beta_n & \alpha_n^2 & 0 & \cdots & 0 \\ \beta_1^2 & 2\alpha_1\beta_1 & \cdots & \beta_n^2 & 2\alpha_n\beta_n & 0 & \cdots & 0 \\ 0 & \beta_1^2 & \cdots & 0 & \beta_n^2 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & 0 & \alpha_1^2 & \cdots & \alpha_n^2 \\ 0 & 0 & \cdots & 0 & 0 & 2\alpha_1\beta_1 & \cdots & 2\alpha_n\beta_n \\ 0 & 0 & \cdots & 0 & 0 & \beta_1 & \cdots & \beta_n \end{pmatrix}.$$

Let $\mathbb{P}_k^*$ denote all bivariate homogeneous polynomials of exact degree k. Then $\mathbb{P}_k = \{0\} \cup \mathbb{P}_0^* \cup \cdots \cup \mathbb{P}_k^*$. We have the following detailed result on the solution space V_n and its dimension by expanding Eq. (2) directly.

Lemma 1. *Let vector space V_n be defined in Eq. (1), and $s = [\frac{\mu+1}{n-1}]$.*

(1) *If $n > \frac{k+1}{k-\mu}$, then $s = 0, 1, \ldots, k-\mu-1$,*

$$V_n \subset \mathbb{P}_s^* \cup \cdots \cup \mathbb{P}_{k-\mu-1}^* \cup \{0\},$$

and

$$d_k^\mu(n) = \sum_{j=0}^{k-\mu-1-s} (n(k-\mu-j) - (k+1-j)) = (n - c_k^\mu(s)) q_k^\mu(s),$$

where $c_k^\mu(s) = \frac{k+\mu+s+3}{k-\mu+s+1}$, and $q_k^\mu(s) = \sum_{i=s}^{k-\mu-1} (i+1) = \frac{1}{2}(k-\mu+s+1)(k-\mu-s)$ denotes the maximum degree of freedom of the smoothing cofactors $q_1, \ldots, q_n \in V_n$.

(2) *If $n \le \frac{k+1}{k-\mu}$, then $s \ge k-\mu$,*

$$V_n = \{0\}, \qquad q_k^\mu(k-\mu) = 0, \quad \text{and} \quad d_k^\mu(n) = 0.$$

(3) *If* $n > \mu + 2$, *then* $s = 0$,

$$V_n \subset \mathbb{P}_0^* \cup \cdots \cup \mathbb{P}_{k-\mu-1}^* \cup \{0\}, \qquad q_k^\mu := q_k^\mu(0) = \frac{1}{2}(k-\mu+1)(k-\mu),$$

and

$$\begin{cases} d_k^\mu(n) = n\binom{k-\mu+1}{2} - \binom{k+2}{2} + \binom{\mu+2}{2} = (n - c_k^\mu)q_k^\mu, \\ d_k^\mu(n+p) = d_k^\mu(n) + p \cdot q_k^\mu, \quad p = 1, 2, \ldots, \end{cases}$$

where $c_k^\mu := c_k^\mu(0) = \frac{k+\mu+3}{k-\mu+1} \leq \mu + 2$.

(4) *If* $n = \mu + 2$, *then* $s = 1$,

$$d_k^\mu(n) = (n - c_k^\mu(1))q_k^\mu(1) = (n - c_k^\mu(0))q_k^\mu(0),$$
$$(n + p - c_k^\mu(0))q_k^\mu(0) = (n - c_k^\mu(1))q_k^\mu(1) + p \cdot q_k^\mu(0), \quad p = 1, 2, \ldots.$$

Hence, if and only if $n \geq \mu + 2$,

$$d_k^\mu(n+p) = d_k^\mu(n) + p \cdot q_k^\mu, \quad p = 1, 2, \ldots.$$

We call a partition with only one interior vertex as a star partition (denoted by Δ_s). It is a special quasi-crosscut partition, then the stability of the dimension of arbitrary spline space on Δ_s can be obtained as follows.

Lemma 2. *For any star partition Δ_s including N interior grid edges, the dimension of the spline space $S_k^\mu(\Delta_s)$ is stable if and only if $N \geq 2\mu + 3$, and*

$$\dim S_k^\mu(\Delta_s) = \binom{k+2}{2} + d_k^\mu(N). \tag{5}$$

Proof. By Theorem 2,

$$\dim S_k^\mu(\Delta_s) = \binom{k+2}{2} + L_1\binom{k-\mu+1}{2} + d_k^\mu(N - L_1),$$

where L_1 denotes the number of crosscuts in Δ_s, $L_1 = 0, 1, \ldots, [\frac{N}{2}]$. The dimension is stable if and only if it is fixed regardless that L_1 varies, i.e.,

$$d_k^\mu(N) = d_k^\mu(N - L_1) + L_1\binom{k-\mu+1}{2}.$$

By Lemma 1, the equivalent condition is $N - L_1 \geq \mu + 2$, i.e. $N \geq 2\mu + 3$. □

By Lemma 2, the necessary condition for the stability of the dimension of arbitrary spline space can be obtained apparently.

Theorem 3. *Given an arbitrary partition Δ including at least one interior vertex, if the dimension of the spline space $S_k^\mu(\Delta)$ is stable, then all the degree of the vertices $N_i \geq 2\mu + 3$.*

The sufficiency of $N_i \geq 2\mu + 3$ for stability of the dimension will be discussed in the following section.

3. The directed partitions and the dimensions of the spline spaces

For a general partition Δ except Δ_c and Δ_{qc}, the dimensions of the spline spaces $S_k^\mu(\Delta)$ are more complicated. We try to discuss the dimensions and their stability with some constraints by generalizing crosscut partition and quasi-crosscut partition. We need some concepts and definitions as follows.

We call a chain of grid lines connected consecutively without self-intersection as a piecewise grid line, or a generalized grid line (GGL). It is clear that two grid edges connected consecutively with same slope must belong to one generalized grid line. They can be classified into the following three kinds by their endpoints.

Definition 2. For each generalized grid line, there are three cases:

(a) *generalized crosscut line (GCL)*: both of its endpoints lie on ∂D.
(b) *generalized ray line (GRL)*: only one of its endpoints lies on ∂D.
(c) *generalized truncated-line (GTL)*: neither of its endpoints lie on ∂D.

In fact, the above three kinds of generalized grid lines can be transformed by crosscuts, rays, and T-lines topologically. Furthermore, we call a partition as generalized crosscut partition (denoted by Δ_{gc}) or generalized quasi-crosscut partition (denoted by Δ_{gqc}), if it can be transformed from a crosscut partition or a quasi-crosscut partition topologically.

A partition is called an even partition, if the degree of each interior vertex is even. It is easy to know

Lemma 3. *Any even partition is a generalized crosscut partition.*

Definition 3. A generalized grid line (GGL) is called a directed GGL if it has a direction from one endpoint to the other one. A partition Δ is called a directed partition if all its GGL are directed, denoted by $\overrightarrow{\Delta}$.

According to Definition 2, we have directed GCL, directed GRL and directed GTL, respectively. For each interior vertex A_i in a directed partition, the in-degree N_i^I and the out-degree N_i^O denote the numbers of grid edges directing in and out A_i respectively, then $N_i = N_i^I + N_i^O$, and $c_i \le N_i^I, N_i^O \le e_i$.

Definition 4. An interior vertex A_i is called a source vertex if $N_i^O = e_i$. Especially, a crosscutting vertex A_i is a source vertex since $c_i = N_i^I = N_i^O = e_i$.

For each interior edge connecting two interior vertices A_i and A_j in a directed partition $\overrightarrow{\Delta}$, for example, it can be directed as $\overrightarrow{A_iA_j}$ by the direction of the GGL, which it belongs to. Then $\overrightarrow{A_iA_j}$ is an out-edge as for A_i, and an in-edge as for A_j. By the Smoothing Cofactor-Conformality method, the smoothing cofactor $q_{i,j} \in \mathbb{P}_{k-\mu-1}$ on $\overrightarrow{A_iA_j}$ belongs to two solution spaces corresponding to the conformality condition at A_i and A_j simultaneously. By Lemma 1, there exist two numbers $s_i, s_j \ge 0$ such that $s_i = [\frac{\mu+1}{e_i-1}]$, and $s_j = [\frac{\mu+1}{e_j-1}]$, we have

(a) if there does not exist $\overrightarrow{A_hA_i}$ such that $\overrightarrow{A_hA_iA_j}$ crosscuts at A_i, then $q_{i,j} \in \mathbb{P}^*_{s_i} \cup \cdots \cup \mathbb{P}^*_{k-\mu-1} \cup \{0\}$ as for A_i, then the minimal degree of nonzero $q_{i,j}$ is s_i, we call it as min-out-degree of $q_{i,j}$ (denoted by $\min \deg_O q_{i,j} = s_i$).

(b) if there does not exist $\overrightarrow{A_jA_l}$ such that $\overrightarrow{A_iA_jA_l}$ crosscuts at A_j, then $q_{i,j} \in \mathbb{P}^*_{s_j} \cup \cdots \cup \mathbb{P}^*_{k-\mu-1} \cup \{0\}$ as for A_j, then the minimal degree of nonzero $q_{i,j}$ is s_j, we call it as min-in-degree of $q_{i,j}$ (denoted by $\min \deg_I q_{i,j} = s_j$).

(c) if there exists $\overrightarrow{A_hA_i}$ such that $\overrightarrow{A_hA_iA_j}$ crosscuts at A_i, then $q_{i,j} - q_{h,i} \in \mathbb{P}^*_{s_i} \cup \cdots \cup \mathbb{P}^*_{k-\mu-1} \cup \{0\}$ as for A_i, then $\min \deg_O q_{i,j} = \min\{\min \deg_I q_{h,i}, s_i\}$.

(d) if there exists $\overrightarrow{A_jA_l}$ such that $\overrightarrow{A_iA_jA_l}$ crosscuts at A_j, then $q_{j,l} - q_{i,j} \in \mathbb{P}^*_{s_j} \cup \cdots \cup \mathbb{P}^*_{k-\mu-1} \cup \{0\}$ as for A_j, then $\min \deg_I q_{i,j} = \min \deg_O q_{i,j}$.

Besides, when one of the endpoints of a directed interior edge $\overrightarrow{A_iA_j}$ lies on the boundary of the domain, we have

(a) if A_i lies on ∂D, then the smoothing cofactor $q_{i,j}$ according to $\overrightarrow{A_iA_j}$ does not need to satisfy the conformality condition at A_i, hence, $\min \deg_O q_{i,j} = 0$.

(b) if A_j lies on ∂D, then the smoothing cofactor $q_{i,j}$ according to $\overrightarrow{A_iA_j}$ does not need to satisfy the conformality condition at A_j, hence, $\min \deg_I q_{i,j} = 0$.

Definition 5. An interior edge $\overrightarrow{A_iA_j}$ of a directed partition is compatible with its direction if $\min \deg_O q_{i,j} \ge \min \deg_I q_{i,j}$. A directed partition is called a compatible directed partition if all its interior edges are compatible with their directions accordingly.

Lemma 4. *If a directed interior edge $\overrightarrow{A_iA_j}$ satisfies one of the following cases, then it is compatible with its direction.*

(a) *$\overrightarrow{A_iA_j}$ is a part of a crosscut of the partition, then* $\min \deg_I q_{i,j} = \min \deg_O q_{i,j} = 0$.

(b) *$\overrightarrow{A_iA_j}$ is a part of a directed ray $\overrightarrow{A_{i_1}A_{i_2}\ldots A_{i_t}}$ with A_{i_t} lying on ∂D.*

(c) *$\overrightarrow{A_iA_j}$ is a part of a directed T-line $\overrightarrow{A_{i_1}A_{i_2}\ldots A_{i_t}}$ with* $e_{i_t} = \max\{e_{i_1}, e_{i_2}, \ldots, e_{i_t}\}$.

(d) *$\overrightarrow{A_iA_j}$ is a part of a directed T-line $\overrightarrow{A_{i_1}A_{i_2}\ldots A_{i_t}}$ with* $\min\{e_{i_1}, e_{i_t}\} > \mu + 2$.

By the aid of the directions of a compatible directed partition $\overrightarrow{\Delta}$ including at least one source vertex, the dimension of the spline space defined on Δ can be calculated with some constraints depending on the degree, the smoothness order and the structure of the partition. The main result of the paper is as follows.

Theorem 4. *Given a compatible directed partition $\overrightarrow{\Delta}$ of a domain D in $\mathbb{R}^2$, with L crosscuts and V_I interior vertices $A_1, A_2, \ldots, A_{V_I}$ including at least one source vertex, such that e_i grid segments of different slopes intersect at A_i, $s_i = [\frac{\mu+1}{e_i-1}]$. If $N_i^O \geq c_k^\mu(s_i) = \frac{k+\mu+s_i+3}{k-\mu+s_i+1}$ when $N_i^O < e_i$, $i = 1, 2, \ldots, V_I$, then*

$$\dim S_k^\mu(\Delta) = \binom{k+2}{2} + L\binom{k-\mu+1}{2} + \sum_{i=1}^{V_I}(N_i^O - c_k^\mu(s_i))q_k^\mu(s_i). \tag{6}$$

Proof. By the Smoothing Cofactor-Conformality method, the dimension of the spline space $S_k^\mu(\Delta)$ is determined by the dimension of the solution space of the global conformality systems. Since $\overrightarrow{\Delta}$ is a compatible directed partition with at least one source vertex, the global conformality systems can be solved by each conformality equation at the corresponding interior vertex sequentially along the directions from the source vertices. For each interior vertex, let the smoothing cofactors on the grid edges directing in the vertex be determined by the former vertices. Then the dimension is added by the freedom of the rest smoothing cofactors on the grid edges directing out the vertex. That is

$$\begin{aligned}\dim S_k^\mu(\Delta) &= \binom{k+2}{2} + \sum_{i=1}^{V_I}(c_i \cdot q_k^\mu(s_i) + d_k^\mu(e_i) - N_i^I \cdot q_k^\mu(s_i)) \\ &= \binom{k+2}{2} + L\binom{k-\mu+1}{2} + \sum_{i=1}^{V_I}(N_i^O - c_k^\mu(s_i))q_k^\mu(s_i).\end{aligned}$$

By Lemma 1, the compatibility of the directions and constraints $c_i \cdot q_k^\mu(s_i) + d_k^\mu(e_i) \geq N_i^I \cdot q_k^\mu(s_i)$, i.e, $N_i^O \geq c_k^\mu(s_i)$ are sufficient to guarantee all the smoothing cofactors on the grid edges directing in each vertex are free in the corresponding conformality system. □

Corollary 1. *For an even partition Δ including L crosscuts, E_I interior edges and V_I interior vertices, such that e grid edges of different slopes intersect at each vertex, $s = [\frac{\mu+1}{e-1}]$. If*

$$\begin{cases} k \geq \dfrac{(e+2)\mu+6-e}{e-2} - \left[\dfrac{\mu+1}{e-1}\right], & e \text{ is even,} \\ k \geq \dfrac{(e+3)\mu+5-e}{e-1} - \left[\dfrac{\mu+1}{e-1}\right], & e \text{ is odd,} \end{cases}$$

then the dimension of $S_k^\mu(\Delta)$ is weakly stable and

$$\dim S_k^\mu(\Delta) = \binom{k+2}{2} + L\binom{k-\mu+1}{2} + (E_I - L - V_I c_k^\mu(s))q_k^\mu(s). \tag{7}$$

Proof. By Lemma 3, Δ is generalized crosscut partition, then we can choose any interior vertex as the source vertex, and assign directions to each interior edge according to the source vertex such that $N_i^O \geq [\frac{e+1}{2}]$ for each vertex, $i = 1, 2, \ldots, V_I$. By Lemma 4, the partition is compatible with the directions since $e_i = e$, for $i = 1, 2, \ldots, V_I$. By the condition, $N_i^O \geq [\frac{e+1}{2}] \geq c_k^\mu(s_i) = c_k^\mu(s)$, $i = 1, 2, \ldots, V_I$. Hence, the dimension of the spline space $S_k^\mu(\Delta)$ is obtained by Theorem 4 with the equality $\sum_{i=1}^{V_I} N_i^O = E_I - L$. □

By Lemma 1 and Theorem 4, we have the following result.

Theorem 5. *Let Δ_{gc} (or Δ_{gqc}) be a generalized (quasi-)crosscut partition of domain D with E_I interior edges and V_I interior grid vertices $A_1, \ldots, A_{V_I}$. If and only if for each A_i, the degree $N_i \geq 2\mu + 3$ $(i = 1, \ldots, V_I)$, then the dimension of the spline space $S_k^\mu(\Delta_{gc})$ (or $S_k^\mu(\Delta_{gqc})$), $0 \leq \mu \leq k-1$, is stable and*

$$\dim S_k^\mu(\Delta_{gc}/\Delta_{gqc}) = \binom{k+2}{2} + E_I\binom{k-\mu+1}{2} - V_I\left(\binom{k+2}{2} - \binom{\mu+2}{2}\right). \tag{8}$$

Proof. For the case of generalized crosscut partition Δ_{gc} with L crosscuts, then N_i is even, $N_i \geq 2\mu+4$ and $e_i \geq \frac{N_i}{2} \geq \mu+2$. If $e_i = \frac{N_i}{2} = \mu+2$ then $s_i = 1$ and A_i is a crosscutting vertex. Else $e_i > \mu+2$, then $s_i = 0$ and by Lemma 4 all the edges are compatible with arbitrary directions. Hence, we can choose at least one interior vertex as the source vertex, and assign

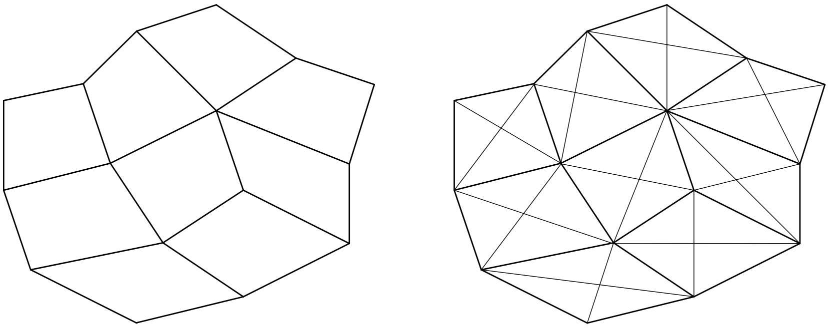


Fig. 2. A quadrangulation and the corresponding triangulated quadrangulation.

directions to each interior edge such that $N_i^O \geq \mu + 2 \geq c_k^\mu(0)$ for the rest vertices except the source vertices. Suppose that there are V_1 crosscutting vertices with $N_i^O = \mu + 2\,(i = 1, \ldots, V_1)$, by Lemma 1 and Theorem 4, we have

$$\begin{aligned}
\dim S_k^\mu(\Delta_{qc}) &= \binom{k+2}{2} + \sum_{i=1}^{V_I}(c_i \cdot q_k^\mu(s_i) + d_k^\mu(e_i) - N_i^I \cdot q_k^\mu(s_i)) \\
&= \binom{k+2}{2} + L \cdot q_k^\mu(0) + \sum_{i=1}^{V_1}(N_i^O - c_k^\mu(1))q_k^\mu(1) + \sum_{i=V_1+1}^{V_I}(N_i^O - c_k^\mu(0))q_k^\mu(0) \\
&= \binom{k+2}{2} + L \cdot q_k^\mu(0) + \sum_{i=1}^{V_I}(N_i^O - c_k^\mu(0))q_k^\mu(0) \\
&= \binom{k+2}{2} + (E_I - V_I \cdot c_k^\mu)q_k^\mu \\
&= \binom{k+2}{2} + E_I\binom{k-\mu+1}{2} - V_I\left(\binom{k+2}{2} - \binom{\mu+2}{2}\right).
\end{aligned}$$

The case of generalized quasi-crosscut partition Δ_{gqc} can be proved similarly. □

If the partition is a triangulation, then the dimension is also the lower bound of dimension presented by [1]. Especially, as an easy application of Theorem 4, we have the following result.

Theorem 6. *For any $k \geq 1$ and any triangulation Δ,*

$$\dim S_k^0(\Delta) = \frac{(k+2)(k+1)}{2} + \frac{(k+1)k}{2}E_I - \frac{(k+3)k}{2}V_I. \tag{9}$$

Proof. For any triangulation Δ with L crosscuts, then $N_i \geq 3$ and $e_i \geq 2$. If $e_i = 2$ then $s_i = 1$ and A_i is a crosscutting vertex. Else $e_i > 2$, then $s_i = 0$ and by Lemma 4 all the edges are compatible with arbitrary directions. Hence, we can choose at least one interior vertex as the source vertex, and assign directions to each interior edge such that $N_i^O \geq 2 \geq c_k^0(0) = \frac{k+3}{k+1}$ for the rest vertices except the source vertices. Similarly to the proof of Theorem 5, by Lemma 1 and Theorem 4, we have

$$\begin{aligned}
\dim S_k^0(\Delta) &= \binom{k+2}{2} + L \cdot q_k^0 + \sum_{i=1}^{V_I}(N_i^O - c_k^0)q_k^0 \\
&= \frac{(k+2)(k+1)}{2} + \frac{(k+1)k}{2}E_I - \frac{(k+3)k}{2}V_I. \quad \square
\end{aligned}$$

Theorem 5 can be applied to some special triangulation partition, such as triangulated quadrangulation, non-uniform type-1 and type-2 triangulations.

Suppose that $\Diamond$ is a nondegenerate convex quadrangulation of a polygonal domain D in $\mathbb{R}^2$. Let Δ_{tq} be the triangulation of $\Diamond$ generated by adjoining both diagonals of each quadrangle, as shown in Fig. 2.

Since there are at least three quadrangles adjacent at each interior vertex, the degree of the interior vertex except the intersections of the two diagonals of each quadrangle is at least 6. By $6 \geq 2\mu + 3$, then $\mu \leq 1$, the dimension of the spline space on Δ_{tq} is given by Eq. (8).

Theorem 7. *For a triangulated quadrangulation Δ_{tq} with Q quadrangles, E_I interior edges and V_I interior vertices except the intersections of the two diagonals of each quadrangle. If and only if $N_i \geq 2\mu + 3$ ($i = 1, \ldots, V_I$), then*

$$\dim S_k^\mu(\Delta_{tq}) = \binom{k+2}{2} + E_I \binom{k-\mu+1}{2} - V_I \left(\binom{k+2}{2} - \binom{\mu+2}{2} \right) + Q \cdot d_k^\mu(2). \tag{10}$$

A non-uniform type-1 triangulation (denoted by $\Delta_{mn}^{(1)}$) is generated by adjoining one directional diagonal of each rectangle in a non-uniform rectangular partition with mn rectangles. Then $N_i = 6, i = 1, \ldots, V_I$, by Theorem 5 we have the following theorem.

Theorem 8. *If and only if $\mu \leq 1$, the dimension of the spline space on non-uniform type-1 triangulation is stable and*

$$\dim S_k^\mu(\Delta_{mn}^{(1)}) = \binom{k+2}{2} + (3mn - m - n) \binom{k-\mu+1}{2} - (m-1)(n-1) \left(\binom{k+2}{2} - \binom{\mu+2}{2} \right). \tag{11}$$

This result is equivalent to the dimension of the spline space on uniform type-1 triangulation as $\mu \leq 1$.

A non-uniform type-2 triangulation (denoted by $\Delta_{mn}^{(2)}$) is generated by adjoining both diagonals of each rectangle in a non-uniform rectangular partition. It is a special triangulated quadrangulation. By Theorem 7, $N_i = 8, i = 1, \ldots, V_I$, then we have the following theorem.

Theorem 9. *If and only if $\mu \leq 2$, the dimension of the spline space on non-uniform type-2 triangulation is stable and*

$$\begin{aligned} \dim S_k^\mu(\Delta_{mn}^{(2)}) = {} & \binom{k+2}{2} + (4mn - m - n) \binom{k-\mu+1}{2} \\ & - (m-1)(n-1) \left(\binom{k+2}{2} - \binom{\mu+2}{2} \right) + mn \binom{k-2\mu}{2}. \end{aligned} \tag{12}$$

This result is equivalent to the dimension of the spline space on uniform type-2 triangulation as $\mu \leq 2$.

4. Discussion on the spline space on the Morgan–Scott partition

As presented in Section 2, the necessary and sufficient condition for dimensions of spline spaces on (quasi-)crosscut partition without singularity is the inequality on the degree of each interior grid vertex and the smoothness order of the splines, i.e., $N_i \geq 2\mu + 3$. In this section, we discuss the dimensions of the spline spaces with corresponding constraints by Theorem 4 when some $N_i < 2\mu + 3$. We take the spline spaces on the Morgan–Scott partition as an example for the application of the dimension formula (6).

Example. The spline space on the Morgan–Scott partition in difference cases (Fig. 3(a)–(i)).

(a) For a general MS partition Δ_{ms0} with $\{e_1, e_2, e_3\} = \{4, 4, 4\}$, we have $s_1 = s_2 = s_3 = [\frac{\mu+1}{3}]$.

(b) For a corresponding compatible directed MS partition $\overrightarrow{\Delta}_{ms0}$ with $\{N_1^O, N_2^O, N_3^O\} = \{4, 3, 2\}$, A_1 is a source vertex. By Theorem 4, when $N_i^O \geq c_k^\mu(s_i)$ ($i = 2, 3$), i.e., $k \geq 3\mu + 1 - [\frac{\mu+1}{3}]$,

$$\dim S_k^\mu(\Delta_{ms0}) = \binom{k+2}{2} + (9 - 3c_k^\mu(s))q_k^\mu(s), \quad \text{where } s = \left[\frac{\mu+1}{3}\right]. \tag{13}$$

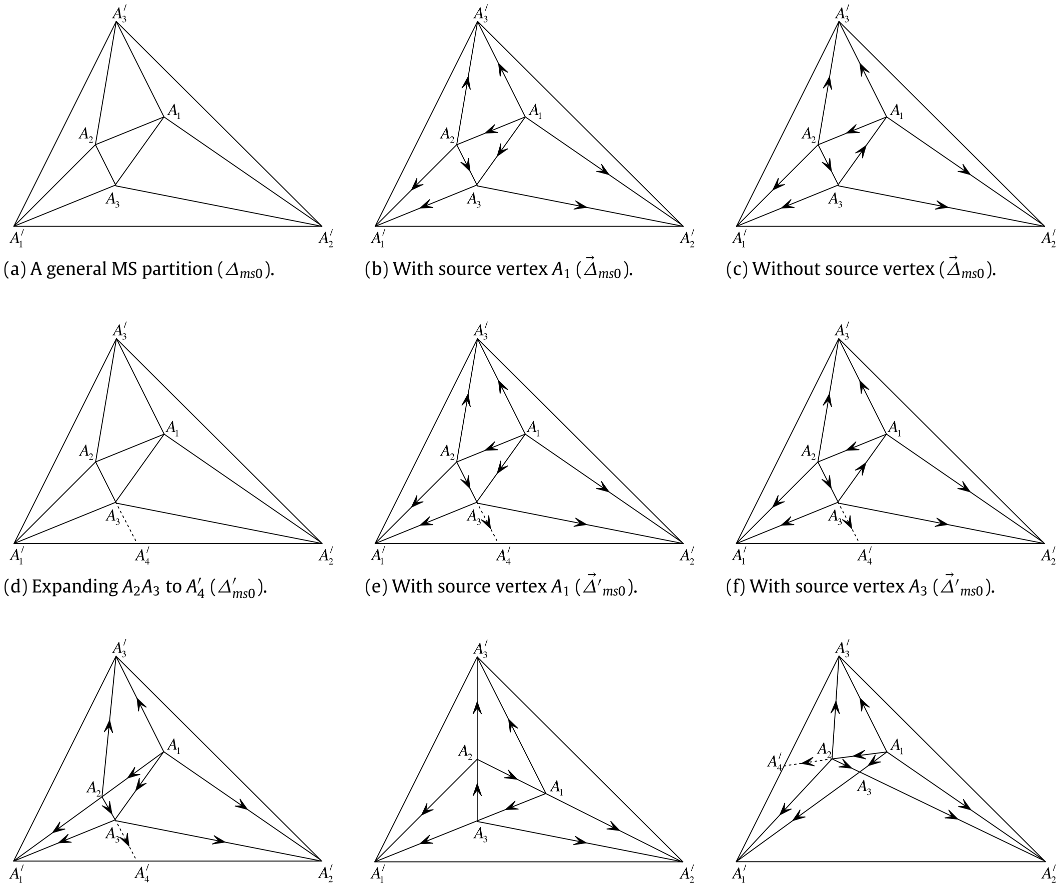
(c) For another corresponding compatible directed MS partition $\overrightarrow{\Delta}_{ms0}$ with $\{N_1^O, N_2^O, N_3^O\} = \{3, 3, 3\}$, there is no source vertex. In order to use the dimension formula (6) with source vertex, we make the following modification on the partition.

(d) Transform Δ_{ms0} to Δ'_{ms0} by expanding A_2A_3 to A'_4, with $\{\hat{e}_1, \hat{e}_2, \hat{e}_3\} = \{e_1, e_2, e_3\}$, and $\{\hat{s}_1, \hat{s}_2, \hat{s}_3\} = \{s_1, s_2, s_3\}$, then

$$\dim S_k^\mu(\Delta_{ms0}) = \dim S_k^\mu(\Delta'_{ms0}) - \dim q_{34'},$$

where $\dim q_{34'}$ denotes the degree of freedom of the smoothing cofactor $q_{34'}$ on $A_3A'_4$ in the global conformality systems of $S_k^\mu(\Delta'_{ms0})$.

(e) A corresponding directed MS partition $\overrightarrow{\Delta'}_{ms0}$ with $\{\hat{N}_1^O, \hat{N}_2^O, \hat{N}_3^O\} = \{4, 3, 3\}$, and A_1 is a source vertex.



(a) A general MS partition (Δ_{ms0}). (b) With source vertex A_1 ($\vec{\Delta}_{ms0}$). (c) Without source vertex ($\vec{\Delta}_{ms0}$).

(d) Expanding A_2A_3 to A_4' (Δ'_{ms0}). (e) With source vertex A_1 ($\vec{\Delta}'_{ms0}$). (f) With source vertex A_3 ($\vec{\Delta}'_{ms0}$).

(g) With one ray A_1A_1' (Δ_{ms1}). (h) Quasi-crosscut partition (Δ_{ms2}). (i) With two intersecting rays A_1A_1' and A_2A_2' (Δ_{ms3}).

Fig. 3. Several compatible directed MS partitions and the corresponding modifications.

(f) Another corresponding directed MS partition $\overrightarrow{\Delta'}_{ms0}$ with $\{\hat{N}_1^O, \hat{N}_2^O, \hat{N}_3^O\} = \{3, 3, 4\}$, and A_3 is a source vertex. It is the same as the case (e).

By Theorem 4, when $\hat{N}_i^O \ge c_k^\mu(\hat{s}_i)$ $(i = 1, 2)$, i.e., $k \ge 2\mu - [\frac{\mu+1}{3}]$,

$$\dim S_k^\mu(\Delta'_{ms0}) = \binom{k+2}{2} + (10 - 3c_k^\mu(s))q_k^\mu(s), \quad \text{where } s = \left[\frac{\mu+1}{3}\right].$$

Since $0 \le \dim q_{34'} \le q_k^\mu(\hat{s}_3)$, we obtain the bound of the dimension of $S_k^\mu(\Delta_{ms0})$,

$$\binom{k+2}{2} + (9 - 3c_k^\mu(s))q_k^\mu(s) \le \dim S_k^\mu(\Delta_{ms0}) \le \binom{k+2}{2} + (10 - 3c_k^\mu(s))q_k^\mu(s). \tag{14}$$

Note that the constraint $k \ge 2\mu - [\frac{\mu+1}{3}]$ is weaker than those in the case (b).

(g) For a kind of special MS partition Δ_{ms1} with one ray A_1A_1', we have $\{e_1, e_2, e_3\} = \{4, 3, 4\}$, and $s_1 = s_3 = [\frac{\mu+1}{3}] \le s_2 = [\frac{\mu+1}{2}]$. As shown in the figure, $\{N_1^O, N_2^O, N_3^O\} = \{4, 3, 2\}$, A_1 and A_2 are source vertices, then by Theorem 4, when $N_3^O \ge c_k^\mu(s_3)$, i.e., $k \ge 3\mu + 1 - [\frac{\mu+1}{3}]$,

$$\dim S_k^\mu(\Delta_{ms1}) = \binom{k+2}{2} + \sum_{i=1}^{3}(N_i^O - c_k^\mu(s_i))q_k^\mu(s_i). \tag{15}$$

In addition, by expanding A_2A_3 to A'_4, then $\{\hat{N}_1^O, \hat{N}_2^O, \hat{N}_3^O\} = \{4, 3, 3\}$, when $\hat{N}_3^O \geq c_k^\mu(s_3)$, i.e., $k \geq 2\mu - [\frac{\mu+1}{3}]$,

$$\dim S_k^\mu(\Delta'_{ms1}) = \binom{k+2}{2} + \sum_{i=1}^{3}(\hat{N}_i^O - c_k^\mu(s_i))q_k^\mu(s_i),$$

$$\binom{k+2}{2} + \sum_{i=1}^{3}(N_i^O - c_k^\mu(s_i))q_k^\mu(s_i) \leq \dim S_k^\mu(\Delta_{ms1}) \leq \binom{k+2}{2} + \sum_{i=1}^{3}(\hat{N}_i^O - c_k^\mu(s_i))q_k^\mu(s_i). \tag{16}$$

(h) For a kind of special MS partition Δ_{ms2} with three rays $A_iA'_i$ $(i = 1, 2, 3)$, we have $\{e_1, e_2, e_3\} = \{3, 3, 3\}$, and $s_1 = s_2 = s_3 = [\frac{\mu+1}{2}]$. As shown in the figure, $\{N_1^O, N_2^O, N_3^O\} = \{3, 3, 3\}$, A_1, A_2 and A_3 are source vertices. In fact, Δ_{ms2} are quasi-crosscut partitions. Let $s = [\frac{\mu+1}{2}]$. Then by Theorem 4, we have

$$\dim S_k^\mu(\Delta_{ms2}) = \binom{k+2}{2} + (9 - 3c_k^\mu(s))q_k^\mu(s) = \binom{k+2}{2} + 3d_k^\mu(3). \tag{17}$$

(i) For a kind of special MS partition Δ_{ms3} with two intersecting rays $A_1A'_1$ and $A_2A'_2$, we have $\{e_1, e_2, e_3\} = \{4, 4, 2\}$, and $s_1 = s_2 = [\frac{\mu+1}{3}] \leq s_3 = \mu + 1$. As shown in the figure, $\{N_1^O, N_2^O, N_3^O\} = \{4, 3, 2\}$, A_1 and A_3 are source vertices, then by Theorem 4, when $N_2^O \geq c_k^\mu(s_2)$, i.e., $k \geq 2\mu - [\frac{\mu+1}{3}]$,

$$\dim S_k^\mu(\Delta_{ms3}) = \binom{k+2}{2} + \sum_{i=1}^{3}(N_i^O - c_k^\mu(s_i))q_k^\mu(s_i). \tag{18}$$

In another way, by expanding A_1A_2 to A'_4, we have $\{\hat{N}_1^O, \hat{N}_2^O, \hat{N}_3^O\} = \{4, 4, 2\}$, A_1, A_2 A_3 are source vertices, and Δ'_{ms3} are quasi-crosscut partitions, hence

$$\dim S_k^\mu(\Delta'_{ms3}) = \binom{k+2}{2} + \sum_{i=1}^{3}(\hat{N}_i^O - c_k^\mu(s_i))q_k^\mu(s_i) = \binom{k+2}{2} + \sum_{i=1}^{3} d_k^\mu(e_i).$$

Noting that $\dim q_{24'} = \min\{d_k^\mu(e_2), q_k^\mu(s_2)\}$, we have

$$\dim S_k^\mu(\Delta_{ms3}) = \binom{k+2}{2} + \sum_{i=1}^{3} d_k^\mu(e_i) - \min\{d_k^\mu(e_2), q_k^\mu(s_2)\}.$$

Especially, when $k \geq 2\mu - [\frac{\mu+1}{3}]$, then $N_2^O = e_2 - 1 \geq c_k^\mu(s_2)$, and $d_k^\mu(e_2) \geq q_k^\mu(s_2)$, so

$$\dim S_k^\mu(\Delta_{ms3}) = \binom{k+2}{2} + \sum_{i=1}^{3}(N_i^O - c_k^\mu(s_i))q_k^\mu(s_i) = \binom{k+2}{2} + \sum_{i=1}^{3} d_k^\mu(e_i) - q_k^\mu(s_2). \tag{19}$$

We obtained the same dimensions (18) and (19) with the same constraints. By comparison with Eq. (17), the singularity of the dimension of the spline space on the Morgan–Scott partition is demonstrated clearly.

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References

[1] L.L. Schumaker, On the dimension of spaces of piecewise polynomials in two variables, in: W. Schempp, K. Zeller (Eds.), Multivariate Approximation Theory, Birkhauser, Basel, 1979, pp. 396–412.
[2] L.L. Schumaker, Bounds on the dimension of spaces of multivariate piecewise polynomials, Rocky Mountain J. Math. 14 (1984) 251–264.
[3] P. Alfeld, L.L. Schumaker, The dimension of bivariate spline spaces of smoothness r for degree $d \geq 4r + 1$, Constr. Approx. 3 (1987) 189–197.
[4] D. Hong, Spaces of bivariate spline functions over triangulations, Approx. Theory Appl. 7 (1991) 56–75.
[5] J. Morgan, R. Scott, The dimension of the space of C^1 piecewise polynomials (1975) (unpublished manuscript).
[6] X.Q. Shi, The singularity of Morgan–Scott triangulation, Comput. Aided Geom. Design 8 (1991) 201–206.
[7] Jian Song Deng, Yu Yu Feng, Jernej Kozak, A note on the dimension of the bivariate spline space over the Morgan–Scott triangulation, SIAM J. Numer. Anal. 37 (3) (2000) 1021–1028.
[8] D. Diener, Instability in the dimension of spaces of bivariate piecewise polynomials of degree $2r$ and smoothness order r, SIAM J. Numer. Anal. 27 (1990) 543–551.
[9] Zhiqiang Xu, Renhong Wang, The instability degree in the dimension of spaces of bivariate spline, Approx. Theory Appl. 18 (1) (2002) 68–80.
[10] Chong-Jun Li, Ren-Hong Wang, Feng Zhang, Improve on the dimensions of spline spaces on T-mesh, J. Inform. Comput. Sci. 3 (2) (2006) 235–244.
[11] Ren-Hong Wang, Juan Chen, Chong-Jun Li, The dimension of spline space on quasi-rectangular partition, J. Math. Res. Exposition 28 (4) (2008) 745–752.
[12] Ren-Hong Wang, The structural characterization and interpolation for multivariate splines, Acta Math. Sinica 18 (1975) 91–106.
[13] Ren-Hong Wang, Multivariate Spline Functions and Their Applications, Science Press, Kluwer Academic Publishers, Beijing, New York, Dordrecht, Boston, London, 2001.
[14] C.K. Chui, R.H. Wang, Multivariate spline spaces, J. Math. Anal. Appl. 94 (1983) 197–221.